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Generalized conifolds and four dimensional $\mathcal{N} = 1$ superconformal theories

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Abstract: This paper lays groundwork for the detailed study of the non-trivial renormalization group flow connecting supersymmetric fixed points in four dimensions using string theory on AdS spaces. Specifically, we consider D3-branes placed at singularities of Calabi-Yau threefolds which generalize the conifold singularity and have an ADE classification. The $\mathcal{N} = 1$ superconformal theories dictating their low-energy dynamics are infrared fixed points arising from deforming the corresponding ADE $\mathcal{N} = 2$ superconformal field theories by mass terms for adjoint chiral fields. We probe the geometry with a single $D3$-brane and discuss the near-horizon supergravity solution for a large number N of coincident $D3$-branes.

Keywords: D-branes, Supersymmetry and Duality, Duality in Gauge Field Theories, Brane Dynamics in Gauge Theories.

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Contents

1. Introduction

The recently proposed duality [1, 2, 3] between string theory on a space B of constant negative curvature and certain gauge theories which live on the boundary of B provides fascinating possibilities for the study of both sides of the equivalence. The original conjecture [1] identifies type IIB string theory on $AdS_5 \times S^5$ with four-dimensional $\mathcal{N} = 4$ super-Yang-Mills theory with gauge group SU(N). In gauge theory terms, the validity of the supergravity approximation to type IIB string theory depends on having both N and the 't Hooft coupling $g_{YM}^2 N$ large.

The conjecture has been extended [4] to the spaces of the form $AdS_5 \times X^5$, where $X^5 = S^5/\Gamma$, with Γ being a discrete subgroup of SO(6). The corresponding gauge theories have been described in [5]. They have $\mathcal{N} = 2$, 1, or 0 superconformal symmetry according as Γ is a subgroup of SU(2), SU(3), or SU(4) $\approx$ SO(6). The low-energy dynamics of N D3-branes placed at an orbifold singularity of a Calabi-Yau three-fold is described by one of these gauge theories.

In general one could consider string theory on $AdS_5 \times M_5$ where M_5 is an arbitrary Einstein manifold. This Freund-Rubin ansatz is the most general static bosonic near-horizon geometry with only the metric and the self-dual five-form excited. Dimension five is the first where there are infinitely many different Einstein manifolds which are not even locally diffeomorphic, and a natural question to ask is what all the corresponding field theories are. The D-brane origin of the holographic conjecture suggests a two step approach to finding the answer: first find a manifold with an isolated singularity such that the near-horizon geometry in supergravity of a black three-brane located on this singularity is $AdS_5 \times M_5$; then figure out the field theory of D3-branes moving close to that singularity. In practice, we may start with a known singularity, work out from supergravity the near-horizon geometry of black three-branes on the singularity, construct a gauge theory describing D3-branes near the singularity, and consider the result as a holographic dual pair. As a rule, the gauge theory is worked out in the approximation that the D3-branes do not significantly distort the geometry. This approximation is correct in the limit of weak coupling, whereas supergravity is valid at strong coupling. If the gauge theory is superconformal, we may feel confident in extrapolating it to strong coupling so that the comparison with supergravity can be made directly. The "extrapolation" of supergravity down to weak coupling is much harder because it requires the full type IIB string theory in a background with Ramond-Ramond fields.

The first successful example of this approach for a manifold not locally diffeomorphic to S^5 was [6]. There the space $M_5 = T^{1,1} \approx (\mathrm{SU}(2) \times \mathrm{SU}(2)) / \mathrm{U}(1)$ was considered, which is the base of what we will call the A_1 conifold:

$$X^2 + Y^2 + Z^2 + T^2 = 0\,. \tag{1.1}$$

N $D3$-branes which are placed at the conifold singularity are described by an $\mathcal{N} = 1$ superconformal field theory which is a non-trivial infrared fixed point of the renormalization group. While this work was in progress, a further class of examples was worked out in [7] using toric geometry.

The purpose of this paper is to construct holographic dual pairs out of an infinite class of conical singularities. The geometry (1.1) is a fibration of a four-dimensional ALE space of type A_1 over the complex plane; our singular geometries will be fibrations of ALE spaces of arbitrary ADE type, and we will call them ADE conifolds. The field theory constructed in [6] descends by RG flow from the $\mathcal{N} = 2$ $\mathbf{S}^5/\mathbb{Z}_2$ orbifold theory with mass terms for chiral fields added to break the supersymmetry to $\mathcal{N} = 1$; our field theories descend from mass deformations of the general A, D, E type $\mathcal{N} = 2$ orbifold theories. Unlike the A_1 case there is a moduli space of such mass deformations which is isomorphic to the projectivization of the moduli space of the versal deformation of the corresponding singularity. In all cases we will have $\mathcal{N} = 1$ superconformal symmetry, which is one quarter of maximal (eight real supercharges).

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Our ADE conifold geometries are non-compact, but they can all be realized as singularities of compact Calabi-Yau three-folds. Our results are most complete for the A_k conifolds, but on many points we include also the analysis for the D_k and E_k cases.

Section 2 is devoted to the study of D3-branes near the orbifold singularities from which our conifold theories descend. If the D3-branes moving in a given singular geometry are claimed to be described by the infrared limit of a particular gauge theory, then the first thing that should be verified is that this gauge theory specialized to a single D3-brane has for its moduli space precisely the singular geometry in question. We present a formal argument why this should be so for the ADE conifold singularities. For $\Gamma = A_k$ or D_k, we present an explicit construction of the Higgs branch, $\mathbb{C}^2/\Gamma$, in the case where the orbifold is not deformed. In the case where it is deformed, we show how the deformation parameters are related to the periods of the complex two-form. For $\Gamma = A_k$, we show explicitly how the conifold arises from the solution of the F- and D-flatness conditions; for the D_k and E_k cases we fall back on the formal argument presented earlier. Finally, we calculate for $\Gamma = A_k$ the Kahler metric from gauge theory at the classical level, exhibit its cone structure and observe that it is not the Calabi-Yau metric. We then briefly discuss the reasons for that, in agreement with the results of [8].

In section 3 we outline the supergravity side of the dual pair. Writing out explicit metrics for the Einstein spaces seems impossible since Calabi-Yau metrics are not known in closed form for the general ADE conifolds. However, we exploit a natural action of $\mathbb{C}^*$ on the conifold geometry to show that the spectrum of chiral primary operators in the gauge theory is correctly reproduced by the holographic mass-dimension relation. We then proceed with more detailed analysis of the blowup modes and the corresponding AdS supergravity multiplets.

In section 4 we present the realization (in the A_k case) of our gauge theories by using other branes as a background instead of the singular geometry.

In the section 5 we present our conclusions and some conjectures.

2. Gauge theory perspective

In this section we describe the geometry of ALE spaces and present a construction of the CY threefold obtained by fibering ALE space over a one-dimensional base. This construction is obtained by looking at the Higgs branch of the gauge theory on the world-volume of a single $D3$-brane.

2.1 Single D-brane on ALE space

First of all we recall the construction of the gauge theory on the world-volume of a single D-brane placed at the orbifold singularity $\mathbb{C}^2/\Gamma$, where Γ is a discrete subgroup of SU(2) of ADE type. The field theory has $\mathcal{N} = 2$ supersymmetry. Its gauge group is the product:

$$G_1 = \times_{i=0}^{r} \mathrm{U}(n_i)\,, \tag{2.1}$$

where i runs through the set of vertices of extended Dynkin diagram of the corresponding ADE type (see figure 1), or, equivalently, through the set of irreducible representations $\mathbf{r_i}$ of Γ. The label $i = 0$ corresponds to the trivial representation. The number n_i is simply the dimension of $\mathbf{r_i}$. Let $h = \sum_i n_i$. This number coincides with the dual Coxeter number of the corresponding ADE Lie group.

The matter content of our gauge theory is that of a_{ij} bi-fundamental hypermultplets in the representations $(n_i, \bar{n}_j)$, where a_{ij} is determined from the decomposition:

$$\mathbb{C}^2 \otimes \mathbf{r}_i = \oplus_j \mathbb{C}^{a_{ij}} \mathbf{r_j}\,. \tag{2.2}$$

From the $\mathcal{N} = 1$ point of view each pair i, j with $a_{ij} \neq 0$ gives rise to a pair of chiral multiplets, call them (B_{ij}, B_{ji}). B_{ij} is a complex matrix, transforming in the $(n_i, \bar{n}_j)$ of the i'th and j'th gauge groups. The theory has a superpotential:

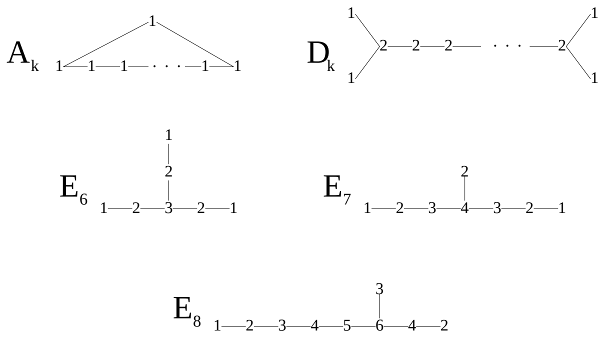

Figure 1: The extended Dynkin diagrams of ADE type, including the indices n_i of each vertex.

$$W = \sum_i \mathrm{Tr}\mu_i\phi_i\,, \tag{2.3}$$

where ϕ_i is the scalar of the i'th vector multiplet and μ_i is the complex moment map. There is some gauge freedom in the choice of explicit expressions for μ_i. Let us introduce an antisymmetric adjacency matrix s_{ij} for the extended Dynkin diagram, such that $s_{ij} = \pm 1$ when i and j are adjacent nodes and the sign, which is part of our gauge choice, indicates a direction on the edge between them. Then we can write

$$\mu_i{}^{\alpha_i}{}_{\beta_i} = \sum_j s_{ij} B_{ij}{}^{\alpha_i}{}_{\gamma_j} B_{ji}{}^{\gamma_j}{}_{\beta_i}\,. \tag{2.4}$$

Here an upper index α_i indicates a fundamental representation of $\mathrm{U}(n_i)$, while a lower index α_j indicates an anti-fundamental representation of $\mathrm{U}(n_i)$. We will suppress these color indices when their contractions are clear from context. There is one relation among the μ_i:

$$\sum_i \mathrm{Tr}\mu_i = 0\,. \tag{2.5}$$

Without breaking $\mathcal{N}=2$ supersymmetry one may introduce complex FI terms which modify superpotential as follows:

$$W \to W - \sum_i \zeta_i \mathrm{Tr}\phi_i \,.$$

The moduli space of vacua (Higgs branch) of the theory with ζ_i's coincides with the complex deformation of the orbifold $\mathbb{C}^2/\Gamma$ into the (smooth for generic ζ_i's) ALE space S_ζ. It can be described as a quotient of the space of solutions of the equations

$$\mu_i = \zeta_i \mathrm{Id}_{n_i} \,, \tag{2.6}$$

by the complexification of the gauge group

$$G_1^c = \times_i \mathrm{GL}_{n_i}(\mathbb{C}) \,.$$

For $\zeta = 0$ the Higgs branch becomes the singular orbifold, and at the singularity the Coulomb branch appears. At the generic point of the Higgs branch there is one massless vector multiplet, which corresponds to the $\mathrm{U}(1)$ subgroup of G_1, which is embedded diagonally into each $\mathrm{U}(n_i)$. Since all the matter is in the bi-fundamentals, it is neutral with respect to this $\mathrm{U}(1)$ subgroup.

It is important to notice that the holomorphic 2-form ω_ζ of S_ζ has periods which depend linearly on ζ (it follows from the complexification of Duistermaat-Heckmann theorem [9, 10]). This observation will be used below.

We would like to list here the equations which describe S_0 for various Γ as hypersurfaces $f_\Gamma = 0$ in the space $\mathbb{C}^3$ with coordinates x, y, z:

$$\begin{aligned} A_k: &\quad f_\Gamma = x^{k+1} + y^2 + z^2 \\ D_k: &\quad f_\Gamma = x^{k-1} + xy^2 + z^2 \\ E_6: &\quad f_\Gamma = x^4 + y^3 + z^2 \\ E_7: &\quad f_\Gamma = x^3 y + y^3 + z^2 \\ E_8: &\quad f_\Gamma = x^5 + y^3 + z^2 \,. \end{aligned} \tag{2.7}$$

The equations $f_\Gamma = 0$ are invariant under a $\mathbb{C}^*$ action, which is specified by giving the weights α, β to the coordinates x, y and z as follows:

Γ	$\alpha = [x]$	$\beta = [y]$	$[z] = h/2$	h
A_k	1	$\frac{k+1}{2}$	$\frac{k+1}{2}$	$k+1$
D_k	2	$k-2$	$k-1$	$2(k-1)$
E_6	3	4	6	12
E_7	4	6	9	18
E_8	6	10	15	30

(2.8)

Notice that $\alpha + \beta = 1 + h/2$.

The complex deformation of the surface S_0 is described by the equation

$$f(x,y;t) + z^2 = 0\,,$$

where t are some coordinates on the base of deformation. There are canonical formulae, listed, say in [11], which represent $f(x,y;t)$ as polynomials in x, y:

$$\begin{aligned}
A_k: &\quad f_\Gamma = P_{k+1}(x) + y^2 + z^2 \\
D_k: &\quad f_\Gamma = x^{k-1} + Q_{k-2}(x) + t_0 y + xy^2 + z^2 \\
E_6: &\quad f_\Gamma = y^3 + Q_2(x)y + P_4(x) + z^2 \\
E_7: &\quad f_\Gamma = y^3 + P_3(x)y + Q_4(x) + z^2 \\
E_8: &\quad f_\Gamma = y^3 + Q_3(x)y + P_5(x) + z^2\,,
\end{aligned} \tag{2.9}$$

where $P_k(x) = x^k + \sum_{\ell=1}^{k-1} t_\ell x^{k-\ell-1}$, $Q_k(x) = \sum_{l=1}^{k+1} t_l x^{k-l+1}$. We now wish to relate the coordinates t_k and ζ_l's. In order to do so we study the periods of the holomorphic two-form,

$$\omega_\zeta = \frac{dx \wedge dy \wedge dz}{df_\Gamma}\,.$$

A_k **case.** In this case the ALE space is a fibration over the x-plane, whose fiber is isomorphic to $\mathbb{C}^*$ for $x \neq x_i$ where x_i are the roots of $P_{k+1}(x_i) = 0, i = 0, \ldots, k$. The form ω_ζ factorizes as: $\omega_\zeta = \frac{dy}{2z} \wedge dx$. To get a non-trivial period of it we choose a one-dimensional contour in the x-plane which connects x_i and x_j for $i \neq j$ and doesn't pass through other x_k's. The fiber over its generic point contains a non-trivial one-cycle, over which the form $dy/2z$ integrates to π (write $y^2 + z^2 = r^2$, $y = r\sin\alpha$, $z = r\cos\alpha$, $0 \leq \alpha < 2\pi$, r is determined by x hence $dy/2z = \frac{1}{2}d\alpha$.) Hence we get

$$[\omega_\zeta]_{ij} = \pi\,(x_i - x_j) = \pi \sum_{m=j}^{i-1} \zeta_m\,, \qquad (i > j)\,. \tag{2.10}$$

The permutations of the roots x_i's act on ζ_i's as the Weyl group of the type A_k.

D_k **case.** In this case the fiber over the point x is the rational curve C_x: $y^2x + t_0 y + R_{k-1}(x) + z^2 = 0$. Consider the discriminant $\Delta(x) = t_0^2 - 4xR_{k-1}(x)$, $R_{k-1}(x) = x^{k-1} + Q_{k-2}(x)$. Let x_i be its roots: $\Delta(x_i) = 0$. For $x \neq x_i$ the rational curve C_x is isomorphic to $\mathbb{C}^*$.

The period of the one-form $dy/2z$ is $\pi/\sqrt{-x}$ for such an x. The two-form is given by: $\omega_\zeta = (dx \wedge dy)/2z$, hence its periods are:

$$[\omega_\zeta]_{ij} = i\pi\left(\sqrt{x_i} - \sqrt{x_j}\right) = \pi \sum_{m=j}^{i-1} \zeta_m\,, \qquad (i > j)\,. \tag{2.11}$$

The branching of the square roots in (2.11) and the permutations of x_i's generate the action of the Weyl group of the type D_k.

E_k **cases.** In these cases the fiber over x is the elliptic curve $y^3+A(x)y+B(x)+z^2 = 0$ and it degenerates over the roots x_i of the discriminant $\Delta(x) = 4A^3(x) + 27B^2(x)$. The periods are given by

$$\int_{x_i}^{x_j} \gamma(x)dx \,,$$

where $\gamma(x) = \oint_{C_x} \frac{dy}{2z}$, C_x is the one-cycle which vanishes both at x_i and x_j. So in this case the identification between the coordinates t_k and ζ_l requires inverting the elliptic functions.

2.2 Single D-brane at the generalized conifold singularity

We now proceed with describing $\mathcal{N} = 1$ theories whose Higgs branch coincides with the non-compact Calabi-Yau manifold Y_Γ with the conifold-like singularity of the following type:

$$F_\Gamma(\phi, x, y, z) \equiv \phi^h f_\Gamma\left(\frac{x}{\phi^\alpha}, \frac{y}{\phi^\beta}; t\right) + z^2 = 0\,. \tag{2.12}$$

The equation (2.12) is homogeneous with respect to the $\mathbb{C}^*$ action described in (2.8) iff the variable ϕ has weight 1.

First of all we need to show that these manifolds have shrunken three-cycles. Let us deform the equation (2.12) to

$$\mu = \phi^h f_\Gamma\left(\frac{x}{\phi^\alpha}, \frac{y}{\phi^\beta}; t\right) + z^2\,.$$

Let us call the non-compact manifold described by this equation as $Y_\Gamma(\mu)$. By construction the manifold $Y_\Gamma(\mu)$ is fibered over ϕ plane with fiber over given ϕ being a particular ALE space $S_{\zeta(\phi,\mu)}$. It is endowed with a holomorphic three-form:

$$\Omega = \frac{d\phi \wedge dx \wedge dy \wedge dz}{dF_\Gamma}\,. \tag{2.13}$$

We are going to show that its periods scale as $\mu^{\frac{2}{h}}$ and therefore vanish in the limit $\mu \to 0$. Indeed, the function F_Γ is homogeneous of degree h with respect to the $\mathbb{C}^*$ action in (2.8). Therefore the form Ω scales as $t^{\frac{2}{h}}$ under the action of the element $t \in \mathbb{C}^*$. Now let us turn to concrete examples of A, D series.

The space $S_{\zeta(\phi,\mu)}$ is fibered over x-plane with the generic fibers being isomorphic to either $\mathbb{C}^*$ in the A, D cases or elliptic curves (with infinity deleted) in the E cases. For given ϕ, μ let us fix a one-dimensional contour connecting x_i and x_j over which the one-cycles vanish. As we vary ϕ these one-dimensional contours span a two-dimensional surface. The nontrivial three-cycle is obtained if we get an interval in ϕ plane which connects two points $\phi^{\pm}_{ij}$ over which the zero-cycle $[x_i] - [x_j]$ shrinks to zero (i.e. the points collide). In the A_k and D_k cases we can be explicit:

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A_k **case.** The conditions on $\phi_{ij}^{\pm}$ are:

$$P_{k+1}\left(\frac{x}{\phi}\right) = \frac{\mu}{\phi^{k+1}}, \qquad P'_{k+1}\left(\frac{x}{\phi}\right) = 0\,.$$

Let $\xi = x/\phi$. Then the period of the three-form Ω reduces to

$$-2\mu^{\frac{2}{k+1}} \oint_{\sigma} \xi \frac{dw}{w^3}\,,$$

where σ is a non-trivial one-cycle on the curve

$$w^{k+1} = P_{k+1}(\xi; t)\,.$$

As $\mu \to 0$ all these periods clearly go to zero.

D_k **case.** Let $\xi = x/\phi^2, \eta = y/\phi$. Consider the curve:

$$w^{k-1} = R_{k-1}(\xi; t) - \frac{1}{4\xi} t_0^2\,.$$

The periods of the three-form Ω reduce to:

$$-\mu^{\frac{1}{k-1}} \oint \sqrt{\xi} \frac{dw}{w^2}\,,$$

$\omega_\zeta = (dx \wedge dy)/2z$, and they also vanish in the limit $\mu \to 0$.

Now we wish to show that the manifold Y_Γ is nothing but the Higgs branch of the $\mathcal{N} = 2$ theory described above perturbed by the superpotential term:

$$W \to W - \sum_i \frac{1}{2} m_i \mathrm{Tr}\phi_i^2\,, \tag{2.14}$$

with the only condition $\sum_i m_i = 0$.

Indeed, let us look at the equations $dW = 0$. By varying with respect to the matter fields we get the condition that ϕ_i must generate a trivial gauge transformation which is only possible when:

$$\phi_i = \phi \mathrm{Id}_{n_i}\,. \tag{2.15}$$

Then, varying with respect to ϕ_i we get:

$$\mu_i = -m_i \phi \mathrm{Id}_{n_i}\,. \tag{2.16}$$

The necessary and sufficient condition for the equations (2.16) to be solvable is precisely $\sum_i n_i m_i = 0$ (it follows from (2.5)). The space of solutions to (2.16) is fibered over the $\phi \neq 0$ plane with the fiber being the (generically) smooth ALE space, corresponding to $\zeta_i = m_i \phi$. Thus the role of the mass vector is to choose the direction in the moduli space of ALE spaces of given ADE type.

In other words, the spaces Y_Γ are constructed as follows. Let $f(x, y)$ be any isolated simple singularity. Let T be the base of its versal deformation. The dimension r of T is called the Milnor index of the singularity. It also coincides with the rank of the corresponding ADE group. The space T has a natural action of the $\mathbb{C}^*$ group, which originates in the $\mathbb{C}^*$ action described in (2.8). The space of orbits of this action is a weighted projective space $W\mathbb{P}_{\{\frac{d_i}{h}\}}$ where d_i are the exponents of f [11]. The space T comes with the canonical bundle Y (called Milnor bundle). Its fiber Y_t over point $t \in T$ is the surface $f(x, y; t) = 0$.

Choose any orbit $t = t(\phi)$ of the $\mathbb{C}^*$ action. Restrict Y onto this orbit. This is our space Y_Γ. It depends on the choice of orbit, that is on the choice of mass parameters m_i.

2.3 Relation to the geometric invariant quotient

In this section we wish to show (in the A and D cases explicitly) that the Higgs branch of the $\mathcal{N} = 2$ theory in the case where all deformation parameters are zero is nothing but the orbifold $\mathbb{C}^2/\Gamma$. To this end we slightly reformulate the solution for the F-flatness conditions. Let $V = \mathbb{C}^\Gamma$ - the space of $\mathbb{C}$-valued functions on the group Γ. This is naturally a representation (called regular representation) of Γ, induced for concreteness by the left action of Γ on itself. For $g \in \Gamma$ let $\gamma(g)$ be the corresponding element of $\mathrm{Hom}(V, V)$. Consider the space of pairs $X_\alpha \in \mathrm{Hom}(V, V)$, $\alpha = 1, 2$ of operators in V which obey two conditions:

$$\begin{aligned} &[X_1, X_2] = 0 \\ &{g_\alpha}^\beta X_\beta = \gamma(g) X_\alpha \gamma(g)^{-1}, \end{aligned} \tag{2.17}$$

where ${g_\alpha}^\beta$ are the matrix elements of g in the two-dimensional representation of Γ. Our space is the quotient of the space of these pairs (X_1, X_2) by the action of the group of gauge transformations (cf. [12]). The latter are the elements of $\mathrm{End}(V)$ which commute with $\gamma(g)$ for all $g \in \Gamma$.

Now let $f(z_1, z_2)$ be any Γ-invariant function on $\mathbb{C}^2$. Consider the matrix $f(X_1, X_2)$. Due to invariance of f we have:

$$\gamma(g) f(X_1, X_2) \gamma(g)^{-1} = f(X_1, X_2), \tag{2.18}$$

for any $g \in \Gamma$.

Now let us look at the A_k, D_k examples in some detail.

A_k case. In this case the solution for (X_1, X_2) is:

$$X_1 = z_1 J_+, \qquad X_2 = z_2 J_-, \tag{2.19}$$

where

$$J_+ = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & & 1 \\ 1 & 0 & 0 & \dots & 0 \end{pmatrix}, \qquad J_- = J_+^t \tag{2.20}$$

in the basis where $\gamma(g)$ is diagonal matrix with entries being $1, \omega, \omega^2, \dots, \omega^k$, $\omega = e^{2\pi i/(k+1)}$. The basic invariants are: $\mathcal{X} = z_1^{k+1}, \mathcal{Y} = z_2^{k+1}, \mathcal{Z} = z_1 z_2$, which obey the equation with A_k singularity:

$$\mathcal{X}\mathcal{Y} = \mathcal{Z}^{k+1}.$$

The corresponding matrix functions are clearly scalars:

$$X_1^{k+1} = \mathcal{X} \cdot \mathrm{Id}, \qquad X_2^{k+1} = \mathcal{Y} \cdot \mathrm{Id}, \qquad X_1 X_2 = \mathcal{Z} \cdot \mathrm{Id}.$$

D_k **case.** In this case the matrices X_1, X_2 have a block diagonal form:

$$X_1 = \begin{pmatrix} z_1 J_+ & 0 \\ 0 & i z_2 J_+ \end{pmatrix}, \qquad X_2 = \begin{pmatrix} z_2 J_- & 0 \\ 0 & i z_1 J_- \end{pmatrix}, \tag{2.21}$$

where the size of $J_\pm$ is $2(k-2) \times 2(k-2)$. The basic invariants here are:

$$\mathcal{X} = z_1^{2(k-2)} + (-)^k z_2^{2(k-2)}, \quad \mathcal{Y} = z_1 z_2 \left(z_1^{2(k-2)} - (-)^k z_2^{2(k-2)} \right), \quad \mathcal{Z} = z_1^2 z_2^2, \tag{2.22}$$

which obey D_k-type equation:

$$\mathcal{Y}^2 = \mathcal{Z}\mathcal{X}^2 - 4(-)^k \mathcal{Z}^{k-1}.$$

It is obvious that

$$\mathcal{X}(X_1, X_2) = \mathcal{X} \cdot \mathrm{Id}, \qquad \mathcal{Y}(X_1, X_2) = \mathcal{Y} \cdot \mathrm{Id}, \qquad \mathcal{Z}(X_1, X_2) = \mathcal{Z} \cdot \mathrm{Id}.$$

The matrices X_1, X_2 provide the most efficient way of making completely explicit the abstract construction of the spaces Y_Γ which we sketched at the end of section 2.1. If we introduce a third matrix Φ with $\gamma(g)\Phi\gamma(g)^{-1} = \Phi$, then the superpotential is

$$W = \mathrm{Tr} \left(\Phi[X_1, X_2] - \frac{1}{2} M \Phi^2 \right). \tag{2.23}$$

The requirement of F-flatness is

$$[\Phi, X_1] = 0, \qquad [\Phi, X_2] = 0, \qquad [X_1, X_2] = M\Phi. \tag{2.24}$$

The first two expressions in (2.24) are satisfied when Φ is a trivial gauge transformation: $\Phi = \phi \mathrm{Id}_{|\Gamma|}$ for some complex number ϕ. Taking the trace of the last equation in (2.24) tells us $\mathrm{Tr}\, M = 0$. The space of solutions to this equation modded out by the complexified gauge group (which implements D-flatness along with gauge invariance) should be the generalized conifold.

A_k **case.** Let us use the notation

$$\mathrm{diag}\,\{x_i\} = \mathrm{diag}\,\{x_i\}_{i=1}^{\Gamma} = \mathrm{diag}\,\{x_1, x_2, \ldots, x_\Gamma\} = \begin{pmatrix} x_1 & 0 & \ldots & 0 \\ 0 & x_2 & \ldots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \ldots & x_\Gamma \end{pmatrix}, \tag{2.25}$$

for diagonal matrices. So for instance $M = \mathrm{diag}\,\{m_i\}$. As a subgroup of SU(2), A_k has as its generating element

$$(g_1)_\alpha{}^\beta = \begin{pmatrix} \omega & 0 \\ 0 & \omega^{-1} \end{pmatrix}_\alpha^{\;\beta}. \tag{2.26}$$

In a basis for the regular representation where $\gamma(g_1) = \mathrm{diag}\left\{1, \omega, \ldots, \omega^k\right\}$, the general solution to $(g_1)_\alpha{}^\beta X_\beta = \gamma(g_1) X_\alpha \gamma(g_1^{-1})$ is $X_1 = \mathrm{diag}\,\{b_{i,i+1}\}\,J_+$, $X_1 = J_-\mathrm{diag}\,\{b_{i+1,i}\}$. We have $X_1X_2 = \mathrm{diag}\,\{b_{i,i+1}b_{i+1,i}\}$ and $X_2X_1 = \mathrm{diag}\{b_{i-1,i}\, b_{i,i-1}\}$. To satisfy $[X_1, X_2] = M\Phi$ we must have $X_1X_2 = x\mathrm{Id}_\Gamma - \Xi\phi$ for some complex number x and a matrix $\Xi = \mathrm{diag}\,\{\xi_i\}$ where $\xi_{i-1} - \xi_i = m_i$. By convention we may take $\xi_i = -\sum_{j=1}^{i} m_i$. Defining the gauge invariant quantities $c_+ = \det X_1$, $c_- = \det X_2$, we recover the A_k conifold equation from

$$c_+c_- = (\det X_1)(\det X_2) = \det(X_1X_2) = \prod_{i=1}^{k+1}(x - \xi_i\phi)\,. \tag{2.27}$$

It is easy to understand this point how the F-flatness conditions eliminate what seems *a priori* to be an excess of gauge-invariant products parameterizing the moduli space. In addition to the products $c_\pm = \prod_i b_{i,i\pm 1}$ which take us all the way around the extended Dynkin diagram, there are $k+1$ products $b_{i,i+1}b_{i+1,i}$. But these may all be expressed in terms of x and ϕ, with (2.27) being the only equation among x, ϕ, and $c_\pm$, so indeed the moduli space has three complex dimensions.

2.4 Construction of the Kahler metric on Y_Γ

Gauge theory gives us an explicit construction of the Kahler metric on the space Y_Γ. Of course, in the case $m = 0$ the metric is exact, while in the $m \neq 0$ case it may be affected by the quantum corrections which lead to the renormalization of the Kahler potential. At any rate we shall describe the metric which one gets by the Kahler quotient construction in the A_k case.

The original space of fields is $\mathcal{B} = \{(\phi_i, b_{i,i+1}; b_{i+1,i})|\; i = 0, \ldots, r\}$, where all fields are complex and we have identified $r + 1 \equiv 0$. We assume that the metric on $\mathcal{B}$ is flat:

$$ds^2 = \sum_{i=0}^{k} d\phi_i d\bar{\phi}_i + db_{i,i+1}d\bar{b}_{i,i+1} + db_{i+1,i}d\bar{b}_{i+1,i}\,. \tag{2.28}$$

We now impose the D- and F-flatness conditions, which means that we restrict the metric (2.28) onto the surface of equations:

$$\begin{aligned} |b_{i,i+1}|^2 - |b_{i+1,i}|^2 + |b_{i,i-1}|^2 - |b_{i-1,i}|^2 &= 0 \\ b_{i,i+1}b_{i+1,i} - b_{i,i-1}b_{i-1,i} &= m_i\phi_i \\ b_{i,i+1}(\phi_i - \phi_{i+1}) &= 0 \\ b_{i+1,i}(\phi_{i+1} - \phi_i) &= 0\,. \end{aligned} \tag{2.29}$$

It is convenient, following [13], to rewrite the flat metric on b's in terms of the coordinates $(\vec{r}_i, \theta_i)$, where

$$\begin{aligned} \vec{r}_i &= (t_i, x_i, \bar{x}_i) \\ |b_{i,i+1}|^2 - |b_{i+1,i}|^2 &= 2t_i \\ b_{i,i+1}b_{i+1,i} &= x_i \\ \frac{b_{i,i+1}}{b_{i+1,i}} &= \left|\frac{b_{i,i+1}}{b_{i+1,i}}\right| e^{2i\theta}\,. \end{aligned} \tag{2.30}$$

We have:

$$db_{i+1,i}d\bar{b}_{i+1,i} + db_{i,i+1}d\bar{b}_{i,i+1} = \frac{1}{r_i}d\vec{r}_i^{\,2} + r_i\left(d\theta_i + \omega_i\right)^2\,, \tag{2.31}$$

where $r_i = |\vec{r}_i|$, and the Dirac connection ω_i obeys:

$$d\omega_i = \star d\frac{1}{r_i}\,,$$

where d is three-dimensional and $\star$ is taken with respect to the flat metric on $\mathbb{R}^3$. The gauge group acts as follows:

$$\theta_i \mapsto \theta_i + \alpha_i - \alpha_{i+1}\,. \tag{2.32}$$

The D, F-flatness conditions imply that:

$$\begin{aligned} \phi_0 = \phi_1 = \cdots = \phi_k &=: \phi \\ t_0 = t_1 = \cdots = t_k &=: t \\ x_i &= x - \xi_i\phi\,, \end{aligned} \tag{2.33}$$

where ξ_i are the complex numbers which are uniquely specified by the following conditions:

$$\xi_i - \xi_{i-1} = -m_i\,, \qquad \sum_i \xi_i = 0\,. \tag{2.34}$$

The projection along the orbits of the gauge group is achieved by taking the orthogonals to the orbit. Formally this is equivalent to the following procedure [13]: replace

$d\theta_i$ by $d\theta + A_i - A_{i+1}$, compute the ds^2 and minimize with respect to A_i. The result is the following metric:

$$ds^2 = V d\vec{r}^2 - U d\phi d\bar{x} - \bar{U} d\bar{\phi} dx + W d\phi d\bar{\phi} + V^{-1} \left(d\theta + A\right)^2 , \tag{2.35}$$

where

$$\begin{aligned} V &= \sum_{i=0}^{k} \frac{1}{\sqrt{t^2 + |x - \xi_i \phi|^2}} \\ U &= \sum_{i=0}^{k} \frac{\xi_i}{\sqrt{t^2 + |x - \xi_i \phi|^2}} \\ W &= \sum_{i=0}^{k} \left(1 + \frac{|\xi_i|^2}{\sqrt{t^2 + |x - \xi_i \phi|^2}} \right) \\ dA &= \star dV , \end{aligned} \tag{2.36}$$

where in the last formula d is taken with respect to $(t, x, \bar{x})$.

2.5 Properties of the metric on Y_Γ

The space Y_Γ comes equipped with the holomorphic three-form. In the A_k case it is given by the formula:

$$\Omega = \frac{dx \wedge d\phi \wedge dc^+}{c^+} , \tag{2.37}$$

where $c^\pm = y \pm iz$. In terms of the coordinates $b_{i,i+1}$ etc., the variables $c^\pm$ are expressed as follows:

$$c^\pm = \prod_{i=0}^{k} b_{i,i\pm1} . \tag{2.38}$$

In solving the D, F-flatness conditions a choice of the gauge for the phases of $b_{i,i\pm1}$ has to be made. We choose:

$$\begin{aligned} b_{i,i+1} &= \left[t + \sqrt{t^2 + |x - \xi_i \phi|^2} \right]^{\frac{1}{2}} e^{\frac{i\theta}{k+1}} \\ b_{i+1,i} &= \left[-t + \sqrt{t^2 + |x - \xi_i \phi|^2} \right]^{\frac{1}{2}} \frac{x - \xi_i \phi}{|x - \xi_i \phi|} e^{-\frac{i\theta}{k+1}} . \end{aligned} \tag{2.39}$$

With this choice of phases the Kahler form ϖ on Y_Γ is written out as follows:

$$\varpi = dt \wedge (d\theta + A) + \frac{i}{2} d\phi \wedge d\bar{\phi} + \frac{i}{4} \sum_{l=0}^{k} \frac{d(x - \xi_l \phi) \wedge d\overline{(x - \xi_l \phi)}}{\sqrt{t^2 + |x - \xi_l \phi|^2}} , \tag{2.40}$$

where

$$A = \frac{1}{2} \sum_{l} \left(\frac{t}{\sqrt{t^2 + |x - \xi_l \phi|^2}} - 1 \right) d\left(\arg(x - \xi_l \phi)\right) . \tag{2.41}$$

Of course, another choice of gauge leads to the gauge transformed A. Now, the holomorphic three-form turns out to have rather simple form:

$$\left(i(d\theta + A) + \frac{1}{2}V dt\right) \wedge dx \wedge d\phi\,. \tag{2.42}$$

From this expression we get the volume form:

$$\Omega \wedge \bar{\Omega} = -iV d\theta \wedge dt \wedge dx \wedge d\bar{x} \wedge d\phi \wedge d\bar{\phi}\,. \tag{2.43}$$

For the Kahler metric to be Ricci-flat it is necessary that:

$$\varpi \wedge \varpi \wedge \varpi = \kappa \Omega \wedge \bar{\Omega} \tag{2.44}$$

for some constant κ. Explicit computation shows that :

$$\varpi^3 = \frac{3}{8}\left(VW - |U|^2\right) dx \wedge d\bar{x} \wedge d\phi \wedge d\bar{\phi} \wedge d\theta \wedge dt \tag{2.45}$$

and (2.44) is not obeyed. Instead, for the Ricci tensor we have:

$$R \equiv R_{i\bar{j}} dz^i \wedge dz^{\bar{j}} = \partial\bar{\partial}\log\left(W - \frac{|U|^2}{V}\right). \tag{2.46}$$

The fact that the metric doesn't come out in the Ricci-flat form may sound troubling. On the other hand it seems that it does not receive quantum loop corrections. The reason is that we study abelian gauge theory and the coupling in this theory is weak in the infrared so all loops must go away.

Hence we are led to believe that, in contrast to $\mathcal{N} = 2$ case, here the geometry as observed by the single $D3$-brane and by the fundamental string is different — unless the assumption (2.28) that the initial Kahler metric was flat is wrong. See [14] for a thorough discussion, and also [8] for more examples of $\mathcal{N} = 1$ orbifolds.

Another problem is that different terms in the formula for the metric (2.35) and the Kahler form (2.40) scale differently under the $\mathbb{R}_+$ which is a part of the R-symmetry (2.8).

2.6 Geometry of the base of the cone

Nevertheless, close to the singularity where $t = |x| = |\phi| = 0$ the term $d\phi \wedge d\bar{\phi}$ can be neglected. As a result of this "RG flow" the metric on the Higgs branch becomes invariant under "RG" action of $\mathbb{R}_+$. Moreover, the Higgs branch becomes a cone over a fivefold M_5 which is in turn a U(1) bundle over four dimensional Kahler manifold B which we now describe in some detail.

Let us think of Y_Γ as the symplectic manifold with the two-form ϖ. It is invariant under the U(1) action $\phi \mapsto \phi e^{i\alpha}, x \mapsto x e^{i\alpha}, \theta \mapsto \theta + \frac{k+1}{2}\alpha$. This action is generated by the Hamiltonian:

$$H = \frac{1}{2}|\phi|^2 + \frac{1}{2}\sum_i \sqrt{t^2 + |x - \xi_i \phi|^2}\,. \tag{2.47}$$

It is easy to show using the metric (2.35) that this action is also free. So we have a fiber bundle:

$$\begin{array}{ccc} \mathrm{U}(1) & \to & Y_\Gamma \\ & & \downarrow \\ & & B\,, \end{array} \tag{2.48}$$

where the manifold B is described as a quotient of the subvariety in Y_Γ defined by the equation $H = \zeta > 0$ by the action of the U(1). The base of the cone M^5 is the level set of the Hamiltonian: $H = \zeta$. To be more precise, consider the following "RG flow": perform the simultaneous rescaling of $t, |x|, |\phi|$ and ζ by the same amount μ and then take the limit $\mu \to 0$. As a result we get the manifold M^5, defined as the hypersurface

$$\sum_i \sqrt{t^2 + |x - \xi_i \phi|^2} = 1\,, \tag{2.49}$$

in the space of t, θ, x, ϕ, with the metric (2.35) where $W \to W - (k+1)$. In the case $k = 1$ the manifold B is the set of pairs of vectors $\vec{x}, \vec{y}$, $\vec{y} \in \mathbb{R}^3, \vec{x} \in \mathbb{R}^2 \subset \mathbb{R}^3$ subject to the condition $|\vec{x} - \vec{y}| + |\vec{x} + \vec{y}| = 1$. It is easy to show that this space is isomorphic to $S^2 \times S^2$ in agreement with the expectations about the conifold geometry (cf. [6]).

In general the manifold B can be described as a complex hypersurface in the weighted projective space $W\mathbb{P}^3_{1,\alpha,\beta,\frac{h}{2}}$ defined by the equation $F_\Gamma = 0$. For example, in the A_1 case we would get the hypersurface in the ordinary $\mathbb{P}^3 = \{(X : Y : Z : T)\}$ defined by the equation $X^2 + Y^2 + Z^2 + T^2 = 0$ that is the quadric $\mathbb{P}^1 \times \mathbb{P}^1$.

3. Supergravity and holography

3.1 Large number N of D-branes at the singularity: Gauge theory

It is clear how to proceed with generalizations: replace the vector spaces $\mathbf{r_i}$ by $R_i = \mathbb{C}^N \otimes \mathbf{r_i}$. The gauge group is now:

$$G_N = \times_i \mathrm{U}(N n_i)\,. \tag{3.1}$$

The matter fields are the a_{ij} hypermultiplets in the bi-fundamental representations: $(Nn_i, \overline{Nn_j})$.

This $\mathcal{N} = 2$ theory describes N coincident $D3$-branes placed at the orbifold point in $\mathbb{C}^2/\Gamma$. We now perturb this theory by adding the mass term:

$$W \to W - \sum_i \frac{1}{2} m_i \mathrm{Tr}\phi_i^2\,. \tag{3.2}$$

In the infrared limit the U(1) factors decouple and one is left with the gauge group

$$\tilde{G}_N = \times_i \mathrm{SU}(N n_i) \tag{3.3}$$

and the effective superpotential:

$$W_{eff} = \sum_{i:\,m_i\neq 0} \frac{1}{2m_i}\mathrm{Tr}\mu_i^2 + \sum_{i:\,m_i=0} \mathrm{Tr}\mu_i\phi_i\,. \tag{3.4}$$

Methods described in [15] can be used to determine the possible anomalous dimensions at the infrared fixed point. Linear constraints on the anomalous dimensions of chiral fields result from setting the NSVZ exact beta functions to zero and demanding that the superpotential be dimension 3. These constraints can always be satisfied in our models by giving the ϕ_i anomalous dimensions of $1/2$ and the B_{ij} anomalous dimensions of $-1/4$. Typically the number of independent constraints is less than the number of independent anomalous dimensions, so there is actually a space of solutions. When all m_i are non-zero, it is straightforward to see from the condition on the superpotential that the dimensions of all gauge-invariant combinations of the B_{ij} are invariant over this space. Thus we can calculate these dimensions at the point where all the B_{ij} have anomalous dimension $-1/4$. This is the result we actually will use in comparisons with supergravity predictions. By continuity we would expect it to continue to hold as some of the m_i are taken to zero (but not all, since the m_i are only defined up to an overall rescaling). We do not have a proof of this, but we would be surprised to find a continuous spectrum of possible dimensions for gauge invariant operators.

We now proceed with showing that UV superpotential yields the moduli space of N $D3$-branes placed at Y_Γ.

Indeed, the discussion of the section related to the geometric invariant theory goes through with the only change that we now tensor $\mathbb{C}^\Gamma$ by the dummy space $\mathbb{C}^N$ which is a trivial representation of the group Γ. As a consequence, the expressions for $X_{1,2}$ now have the following form:

$$X_\alpha^{(N)} = \mathrm{diag}\left(X_\alpha^{(1)}(z_1^1, z_2^1), \ldots, X_\alpha^{(1)}(z_1^N, z_2^N)\right)$$

so they depend on an N-tuple of the parameters z_1, z_2 on which the allowed gauge transformations act as the group Γ does. Turning on the mass matrix M makes z_1, z_2 live in the deformed ALE space. It means that the Higgs branch looks like the N'th symmetric product of the generalized conifold Y_Γ, which is what we expect.

3.2 The supergravity geometry and chiral primaries

Let us start by briefly reminding the reader the supergravity version of D3-branes at an isolated singularity of a six dimensional manifold [16]. Assume the manifold is a Calabi-Yau three-fold and that the metric near the singularity before the addition of D3-branes can be written as

$$ds_6^2 = dr^2 + r^2 g_{\alpha\beta}dx^\alpha dx^\beta\,, \tag{3.5}$$

where $g_{\alpha\beta}$ is an Einstein metric on a five-manifold M_5. When N $D3$-branes are placed at the singularity $r=0$, the supergravity metric is

$$ds^2 = \left(1+\frac{L^4}{r^4}\right)^{-1/2}\left(-dt^2+dx_1^2+dx_2^2+dx_3^2\right)+\left(1+\frac{L^4}{r^4}\right)^{1/2} ds_C^2\,, \tag{3.6}$$

where

$$L^4 = \frac{\sqrt{\pi}}{2}\frac{\kappa N}{\mathrm{Vol}\,M_5}\,. \tag{3.7}$$

Here $2\kappa^2 = 16\pi G = (2\pi)^7 g_s^2 \alpha'^4$ is the gravitational coupling. Supergravity is a good approximation when L is much bigger than the Planck length and the string length.

This pictures applies to the conifolds (2.12) as follows. In (2.8) we specified an action of $\mathbb{C}^*$ action on the conifold geometries. The $\mathbb{R}^+$ part of this action is the dilation symmetry of the cone, $r \to \lambda r$. The $\mathrm{U}(1)$ part of the action is an isometry of M_5, and in the field theory it is realized as the R-symmetry group. From the existence of the Calabi-Yau metric on the conifolds we are learning of the existence of a class of five-dimensional Einstein manifolds. Unlike the coset manifolds constructed in [17], these Einstein manifolds have moduli spaces. To discuss chiral primary operators in following [18] is impractical because we cannot write down the metric explicitly. Fortunately there is a more efficient way, which we will now explain.

A complete set of harmonic functions on the cone can be generated from harmonic functions f which are also eigenfunctions of the operator $r\partial_r$:

$$\begin{aligned} r\partial_r f &= \Delta_f f \\ 2(\bar\partial\bar\partial^* + \bar\partial^*\bar\partial)f &= (dd^*+d^*d)f = \left[\frac{1}{r^5}\partial_r r^5 \partial_r + \frac{1}{r^2}\Box\right] f = 0\,, \end{aligned} \tag{3.8}$$

where we have reserved the symbol $\Box$ to denote the five-dimensional Laplacian on the base of the cone. Together the two conditions in (3.8) imply that $(\Box + E)f = 0$ where $E = \Delta(\Delta+4)$. By considering a complete set of harmonic functions on the cone one can extract the full spectrum of the five-dimensional scalar Laplacian.

The holographic correspondence as worked out in [2, 3] relates on-shell fields in the bulk of spacetime to operators on the boundary. In the present context, following the arguments used in [18], the spectrum of the scalar Laplacian $\Box$ relates to chiral primary operators with dimension $\Delta = -2+\sqrt{4+E}$: exactly the eigenvalue under $r\partial_r$ of the harmonic extension f to the cone! To be more precise, only a subset of the eigenfunctions of $\Box$ correspond to chiral primaries: these are the operators which maximize $\mathrm{U}(1)_R$ charge with dimension held fixed, and the eigenfunctions they are dual to are in fact the ones which extend to holomorphic functions on the cone.

In particular, for the A_k conifolds, we can consider the complex variables $c^\pm$, x, and Φ as holomorphic functions. Near the IR fixed point we know their dimensions from their representations as products of the fields $b_{i,i\pm1}$: $\Delta_{c^\pm} = \frac{3}{4}(k+1)$ and $\Delta_x = \Delta_\Phi = 3/2$. These dimensions are just $3/2$ times the R-charges listed in (2.8). Since these R-charges are determined by the $\mathbb{C}^*$ action on the conifold geometry, of which $r\partial_r$ is one generator, we have shown that the dimensions of $c^\pm$, x, and Φ agree between gauge theory and supergravity, up to an overall normalization. The most direct way to fix that normalization is to note that the metric (3.5) has dimension 2 under dilations of r; hence so does the Kahler form.[1] The cube of the Kahler form is proportional to $\Omega \wedge \bar{\Omega}$, where Ω is the complex three-form. So Ω has dimension 3. Finally, writing out Ω in terms of $c^\pm$, x, and Φ, one can verify that the gauge theory dimensions indeed agree completely with supergravity. Although we have focused on the A_k conifolds the analysis is equally straightforward for the D_k and E_k cases.

Holomorphic functions of the z_i which have a definite eigenvalue under $r\partial_r$ are just polynomials in the z_i which are homogeneous with respect to the weight in (2.8), identified modulo the equation relating the z_i which defines the conifold. We have argued in section 2 that the solution of the F-flatness conditions in the gauge theory, modulo complexified gauge invariance, results in this same conifold. The complexification of the gauge invariance implemented D-flatness. Now, chiral primaries in the gauge theory are constructed from sums of gauge invariant products of chiral superfields, modulo F- and D-flatness conditions, and with definite conformal dimension. It follows that chiral primary operators are in one-to-one correspondence with the homogeneous holomorphic functions on the conifold. This was essentially checked in (34) of [6] for the A_1 case by showing explicitly that F-flatness constraints allowed one to symmetrize A_i and B_j fields separately in a product of the form $A_{i_1}B_{j_1}A_{i_2}B_{j_2}\ldots A_{i_l}B_{j_l}$. The constraints are more complicated in the general ADE case, but the conclusions are the same: because the moduli space (namely the conifold) is parametrized by holomorphic gauge invariant combinations of the matter fields modulo the F-flatness conditions, the chiral primary fields which these combinations represent are precisely the holomorphic functions on the conifold.

It is worth emphasizing that holomorphic functions on the conifold were introduced as a trick to write down efficiently the eigenfunctions of the Laplacian on M_5 which minimized dimension for specified R-charge. The arguments of the previous two paragraphs show with minimal calculation that holography predicts exactly the right dimensions and degeneracies for chiral primary operators in the gauge theory. There are on order Δ^3 chiral primaries with dimension less than Δ. As in the A_1 case [18] (but in contrast to the S^5 case) supergravity predicts in addition on order of Δ^5 non-chiral fields of dimension less than Δ. These come from the non-holomorphic eigenfunctions of the Laplacian on M_5. In the gauge theory they reside in long multi-

[1] We note in passing that an appropriately chosen Kahler potential should also have dimension 2. Written in terms of the complex variables, this amounts to the homogeneity condition

$$\sum_i \Delta_i \left(\frac{\partial}{\partial \log z_i} + \frac{\partial}{\partial \log \bar{z}_i} \right) K = 2K\,, \tag{3.9}$$

where we collectively denote $c^\pm$, x, and Φ by z_i.

plets whose dimensions are not algebraically protected, and as far as we know there is no good understanding for why the dimensions should match the supergravity predictions.

3.3 Blowup Modes and RG flow

If it is indeed true that one can *define* string theory on a manifold which is (at least asymptotically) AdS_5 times a compact manifold through a gauge field theory which lives on the boundary, then we would expect to see reflected in some solution of supergravity the full renormalization group flow from an $\mathcal{N}=2$ theory, deformed by mass terms as in (2.14), to a non-trivial infrared $\mathcal{N}=1$ fixed point with a quartic superpotential. The simplest case would be a supergravity geometry interpolating between $S^5/\mathbb{Z}_2$ and T^{11}. We do not have a complete enough understanding of the Lagrangian of gauged $\mathcal{N}=4$ supergravity in five dimensions to find such solutions explicitly.[2] However, we can at least describe a multiplet which plays a key role.

Blowup modes of the fixed S^1 of S^5/Γ were discussed in [19] (see also the appendix of [20] for a more precise discussion of Kaluza-Klein reduction). Our aim is to indicate how this analysis feeds into the supergravity interpretation of the RG flow. For $\Gamma = A_k$, D_k, or E_k, blowing up the S^1 introduces k independent 2-cycles. The self-dual four-form potential A^+_{MNPQ} on one of these cycles gives rise to an anti-self-dual two-form potential B^-_{MN} in $AdS_5 \times S^1$. The Kaluza-Klein reduction of B^-_{MN} on S^1 leads a tower of fields labelled by the Kaluza-Klein momentum ℓ: at $\ell = 0$ a vector field A_μ satisfying $d * dA = 0$; and for $\ell \neq 0$ an antisymmetric tensor field $A_{\mu\nu}$ satisfying $*dA = -i\ell A$. Both these equations of motion are valid only at the linearized level. Tensor fields which satisfy the latter equation of motion are termed "anti-self-dual" in [21], where (among other things) the superpartners of anti-self-dual antisymmetric tensors and of vectors are worked out using $\mathrm{SU}(2,2|2)$ group theory. The multiplet we will be particularly interested in is the $\ell = 1$ tensor multiplet. The bosonic components, their quantum numbers under the R-symmetry group $\mathrm{SU}(2) \times \mathrm{U}(1)$, and the types of gauge theory operators they are dual to, are as follows:

field	$\mathrm{SU}(2) \times \mathrm{U}(1)$	operator	dimension
scalar	$\mathbf{1}_0$	F^2	4
scalar	$\mathbf{1}_4$	X^2	2
scalar	$\mathbf{3}_2$	$\lambda\lambda$	3
tensor	$\mathbf{1}_2$	$F_{\mu\nu}X$	3

(3.10)

The gauge theory operators we have identified schematically as X^2 in (3.10) can be written more precisely as $\mathrm{Tr}\Phi_i^2 - \mathrm{Tr}\Phi_j^2$. The set of scalar mass terms corresponding

[2] For a globally supersymmetric conformal field theory in four dimensions, $\mathcal{N}=2$ means sixteen real supercharges, which is the same number as in $\mathcal{N}=4$ gauged supergravity in five dimensions. The supergroup organizing the multiplets in both cases is $\mathrm{SU}(2,2|2)$.

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to all the independent 2-cycles of the blown up orbifold form a basis for the mass perturbations introduced in (2.14). The $\lambda\lambda$ operators in (3.10) are the corresponding fermion mass terms which are turned on at the same time to preserve $\mathcal{N} = 1$ supersymmetry.

Part of the analysis of [19] was to determine the dimensions of these operators, listed in (3.10), from the masses of the corresponding modes in the tensor multiplet. The first step in finding a supergravity solution interpolating between the orbifold and conifold geometries should be to turn on the $\lambda\lambda$ and X^2 modes in (3.10) at the linearized level. One would need a concise description of the relevant interactions to extract a solution of the full nonlinear theory, perhaps along the lines of [22], [23]. Unlike the RG flows considered in those papers, the supergravity solution interpolating between the orbifold and the conifold should preserve four real supercharges throughout the flow. At the UV and IR endpoints one should recover sixteen and eight supercharges, respectively.

The relevant part of the multiplet (3.10) which is turned on forms what is called a *spinor* multiplet of $\mathcal{N} = 2$ AdS_5 supergravity [21]. It contains a pair of scalars of U(1) charges ± 1 and a spinor (left- or right- handed) of U(1) charge zero. The U(1) charge assignments can be shifted by the Kaluza-Klein momentum ℓ of the highest spin state. Thus in particular, two right-handed spinor multiplets with $\ell = 0$ and 2, together with an anti-self-dual tensor multiplet of $\mathcal{N} = 2$ with $\ell = 1$, form the anti-self-dual tensor multiplet of $\mathcal{N} = 4$ with $\ell = 1$ that enters into (3.10).

4. Dual constructions with branes

Most of the constructions which we were studying using the geometry or field theory can be redone in the language of branes, along the lines of [24], [25]. The idea is to use the T-duality between the ALE (more precisely multi-Taub-NUT) space and fivebranes. First let us remind the reader of the realization of the $\mathcal{N} = 2$ superconformal theories in this language. Consider N $D3$-branes placed at the orbifold singularity of the ALE space. Let 0123 be the world-volume of the 3-branes, while 6789 are the coordinates of ALE space. Let 6 be the compact direction corresponding to the U(1) isometry of the ALE space. Perform T-duality along the 6'th direction. If the ALE singularity is of A_{k-1} type then we get the Type IIA theory on $\mathbb{R}^{1,8} \times S^1$ with k $NS5$-branes, whose world-volume is 012345 (6 being the coordinate along S^1) and which are located at the same point $\vec{r}$ in the 789-plane. The N $D3$-branes are mapped to the N $D4$-branes which wrap the circle S^1. Their world-volume is 01236. The $NS5$-branes are located at the points $\theta_1, \ldots, \theta_k$. The differences $\theta_i - \theta_{i-1}$ correspond to the fluxes of the $NSNS$ B-field on the Type IIB side. The corresponding RR fluxes become visible if the whole picture is lifted to M-theory, where the circle S^1_6 is promoted to the two-torus $T^2 = S^1_6 \times S^1_{10}$ and the k $NS5$-branes become k $M5$-

branes which are located at the points $z_1, \ldots, z_k$ on the torus T^2. The N $D4$-branes are lifted to N $M5$-branes which wrap the whole of the T^2.

If the ALE singularity was of D type then in addition to $NS5$-branes one finds orientifold plane on Type $\mathbb{I}$A side.

If the $NS5$- (or $M5$-) branes are dislocated in the 789 plane then the corresponding ALE space is resolved. The parameters $\vec{r}_i - \vec{r}_j$ describing the relative positions of the fivebranes in the 789 plane are mapped to the hyperkahler moduli of ALE space. The corresponding process in the field theory is described by turning on the FI terms $\vec{\zeta}$.

The plane 45 transverse to the $D4$-branes has the meaning of the Coulomb branch direction. Let us denote by $\phi = x^4 + ix^5$ the corresponding coordinate. Then the separation of the $D4$-branes in the ϕ direction causes the $NS5$-branes to be frozen at the same point $\vec{r}$ in the 789 plane and vice versa. Of course, this is the familiar picture of the transitions between the Coulomb and Higgs branch with or without FI terms.

Now let us rotate some branes. Let $x^7 + ix^8 = x$ be another holomorphic coordinate. Consider tilting the $NS5$-branes in such a way that for i'th brane its position in 78 plane linearly depends on ϕ:

$$x_i = \xi_i \phi_i \,. \tag{4.1}$$

This configuration of fivebranes preserves supersymmetry (one can think of it as of the fivebrane "wrapping" a holomorphic curve $\prod_i(\zeta - \xi_i\phi) = 0$) and leads to $\mathcal{N} = 1$ gauge theory on the world-volume of $D4$-branes. The tilting makes the scalars in the vector multiplets of $\mathcal{N} = 2$ supersymmetry massive, since the $D4$-branes are no longer free to slide along the $NS5$-branes in the 45 directions. Another way of seeing this is to notice that the condition that FI term is proportional to the scalar in the vector multiplet is identical to the equation (2.16).

Finally, let us consider what happens when we add D5-branes to the stack of N D3-branes on an orbifold singularity. For simplicity we will restrict our attention to an A_{k-1} orbifold singularity which has been resolved by FI terms: the geometry is a direct product of the 4-dimensional ALE space (dimensions 6789), the complex plane (dimensions 45), and flat Minkowski space (dimensions 0123). As remarked above, there are k 2-cycles which sum to zero in homology and through which there are fluxes $\theta_i - \theta_{i-1}$ of the NS B-field. Consider wrapping a D5-brane around one of these cycles, with its other dimensions in the directions 0123. The term linear in B_{NS} in the Wess-Zumino part of the D5-brane action [26],

$$i\mu_5 \int e^{2\pi\alpha' F + B_{NS}} C$$

gives this wrapped D5-brane precisely $(\theta_i - \theta_{i-1})/2\pi$ of a D3-brane charge. This is a special case of the phenomenon of fractional branes and wrapped branes discussed in [12, 27, 28], only here we are allowing arbitrary θ_j rather than taking $\theta_j = 2\pi j/k$.

Upon T-dualizing, the D5-brane becomes an extra D4-brane stretched between the $i-1$'st and i'th NS5-branes.[3] This has the effect of changing the gauge group: it was $\mathrm{SU}(N) \times \cdots \times \mathrm{SU}(N)$, with k factors of $\mathrm{SU}(N)$; now the i'th gauge group becomes $\mathrm{SU}(N+1)$. The supersymmetry is still $\mathcal{N} = 2$, and the hypermultiplets are still in bifundamental representations. The interpretation of $D5$-branes wrapped on a 2-cycle as modifying a gauge theory by incrementing the rank of one gauge group was suggested in [30] based on evidence from anomalous brane creation. The use of T-duality in a perturbative D-brane setting reinforces that interpretation.

It would be nice to T-dualize back from brane realizations of gauge theories to obtain the exact supergravity/string background which are dual to them, similarly to the construction for A_1 case in [25]. Unfortunately, at the moment it does not seem to be very practical.

5. Conclusions and conjectures

So far we described a class of complex threefolds which generalize the ordinary conifold. Our construction is most easily described in the language of the gauge theory on the world-volume of the probe $D3$-brane placed at the singularity of the threefold. We start with the quiver $\mathcal{N} = 2$ gauge theory of the ADE type which corresponds to the manifold which locally looks like $Y_{\Gamma,\mathrm{UV}} = \mathbb{C}^2/\Gamma \times \mathbb{C}$. The manifold $Y_{\Gamma,\mathrm{UV}}$ is a cone over the base $M^5_{\mathrm{UV}} = S^5/\Gamma$. When the large N number of $D3$-branes are placed at the singularity they can no longer be treated as probes. Instead, they change the space-time geometry from that of $\mathbb{R}^{1,3} \times Y_{\Gamma,\mathrm{UV}}$ to $AdS_5 \times M^5_{\mathrm{UV}}$ and there is a flux of RR five-form field through M^5_{UV} which is equal to N. The properties of the string theory propagating in this background are believed to be reflected in those of the superconformal gauge theory which occurs at the origin of the space $Y_{\Gamma,\mathrm{UV}}$ considered as a Higgs branch of the gauge theory on branes.

The $\mathcal{N} = 2$ superconformal theory has a number of interesting deformations. It is known that it has exactly $r+1$ complex marginal deformations corresponding to the couplings of various gauge factors. Their space-time counterparts are the space-time dilaton+axion τ and the fluxes of the RR and NSNS B-fields through the collapsed two-cycles which are fibered over the fixed circle in S^5/Γ [5]. The six-dimensional tensor multiplet which contains these fluxes also contains the parameters of the deformations of the two-cycles themselves (three parameters per cycle). These would correspond to the FI terms in the gauge theory. The $\mathcal{N} = 2$ gauge theory deformed by the generic FI terms flows to the trivial IR fixed point. The space-time interpretation of this fact is that if one first resolves the orbifold $\mathbb{C}^2/\Gamma$ into a smooth space and then places the large number of threebranes at the generic point of it then the near-horizon geometry will be $AdS_5 \times S^5$ as in the absence of any orbifold.

[3]We thank A. Karch for bringing to our attention the reference [29], which includes a similar discussion of this construction.

There are two distinct claims one could make regarding D3-branes located at the Calabi-Yau singularities we have described. The first and simplest is as follows: given a conifold singularity of a particular ADE type, the low-energy theory of D3-branes located at that singularity is the IR fixed point arising from an $\mathcal{N} = 2$ theory deformed by giving masses to the $\mathcal{N} = 1$ chiral multiplets within the $\mathcal{N} = 2$ vector multiplets, as described in section 2 and 3.1. The $\mathcal{N} = 2$ origin of the gauge theory begs the question in what sense one can start with D3-branes at an ADE orbifold singularity and "flow" to the conifold geometry. It was argued in the A_1 case in [6] that there is no topological obstruction to the flow (more specifically that a resolution of $S^5/\mathbb{Z}_2$ has the same topology as T^{11}). We have taken one more step toward describing such a flow by identifying the multiplet of AdS_5 supergravity which includes the blowup modes and observing that in AdS_5 some fields in this multiplet have just the right tachyonic masses to correspond to the scalar and fermion masses involved in deforming the gauge theory. The states in this multiplet arose in the analysis of [19] from the twisted sector localized at the circle on S^5/Γ fixed by the action of Γ. In a nutshell the second claim is that starting with D3-branes at an ADE orbifold singularity, one can turn on fields which in the gauge theory are the mass deformations and in the string theory are twisted sector modes, and obtain a string theory background which tracks the RG flow which takes the gauge theory from its UV fixed point (with $\mathcal{N} = 2$ supersymmetry) to its IR fixed point (with $\mathcal{N} = 1$).

Assuming such a string background exists, what are its properties? It should have the rotational and translational symmetries of four-dimensional Minkowski space, and it should preserve four real supercharges. Five-dimensional supergravity is a valid description of the low-energy dynamics of both ends of the flow (provided we take N sufficiently large and include the matter multiplets arising from the twisted sector of the orbifold), so it seems likely that it is in fact valid throughout the flow. It is not clear whether truncating the theory to a small number of multiplets (as was done in effect in [23] and [22]) is a controlled approximation far from the fixed points. The AdS_5 metric should be recovered at either end of the flow (although with different radii, related to the central charges as in [18]), and in the full ten-dimensional string theory description we expect to see the metric smoothly approach the factorized form $AdS_5 \times M^5_{\mathrm{UV}}$ in the ultraviolet and $AdS_5 \times M^5_{\mathrm{IR}}$ in the infrared. $M^5_{\mathrm{UV}} = S^5/\Gamma$ as above, and M^5_{IR} is the base of the cone described in section 2.6. The total space of the cone is the Calabi-Yau manifold, whose complex structure we described in the previous sections. It is not clear to us what on this cone should play the role of a radial coordinate, dual to scale in the RG flow.

To put it in a single phrase, the two ends of the RG group flow correspond to the two Einstein manifolds, M^5_{UV} and M^5_{IR}.

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